ALGEBRA

List 3.

Matrices. Determinants. Inverse matrices

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1. For the matrices A, B, C given below, which of the matrices: $A{+}B, A{+}C, 2A, AB, BA, AC, CA, A^2, C^2$ are well defined? Compute the matrices which are well defined.

(a)
$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 4 & 3 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix};$$

(b)
$$A = \begin{pmatrix} 2 & 1 \\ -1 & -2 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

(c)
$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

2. Let
$$A = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 1 \\ 3 & 5 \end{pmatrix}$$

Compute AB, and then solve the matrix equations

(a) $AX = C$;

(b) $XA = C$;

(c) $AXB = C$.

3. It is known that
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6.$$

Find the following determinants:

$$\text{(a)} \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}, \quad \text{(b)} \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}, \quad \text{(c)} \begin{vmatrix} g & d & a \\ h & e & b \\ i & f & c \end{vmatrix}, \quad \text{(d)} \begin{vmatrix} 3a & -b & c+4a \\ 3d & -e & f+4d \\ 3g & -h & i+4g \end{vmatrix}.$$

4. Let A be a real 7×7-matrix with $\det A = 2$. Find the determinants of the following matrices: (a) $2A$; (b) $-5A$; (c) $-A^3$; (d) AA^T.

5. Write the Laplace expansions of the given determinants along indicated rows or columns

$$\text{(a)} \begin{pmatrix} 2 & 1 & \mathbf{2} \\ 3 & 2 & \mathbf{1} \\ 4 & 3 & \mathbf{-1} \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 2 & 1 & 3 & 1 \\ \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} \\ 1 & -1 & 1 & 3 \\ 2 & -2 & 1 & -3 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 1 & -3 & 1 & \mathbf{2} \\ 2 & 3 & -2 & \mathbf{1} \\ -2 & 1 & 1 & \mathbf{0} \\ 1 & 4 & 3 & \mathbf{0} \end{pmatrix}.$$

6. Calculate the determinants from the previous problem.

7. Calculate the determinants

$$(a) \begin{pmatrix} 1 & 2 & -3 \\ -2 & 3 & -1 \\ 3 & 2 & 1 \end{pmatrix}, \quad (b) \begin{pmatrix} 2 & 0 & 2 & 3 \\ -1 & 1 & -1 & 1 \\ -2 & 0 & 2 & 0 \\ 5 & -1 & -1 & 1 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 & 0 & 2 & 3 \\ -2 & 0 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ -1 & 0 & 5 & 1 \end{pmatrix}, \quad (d) \begin{pmatrix} 4 & 1 & 3 & 0 \\ 4 & 1 & 0 & 2 \\ 4 & 0 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix},$$

$$(e) \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

8. Calculate the determinants for the matrices which depend on real parameters:

$$(a) \begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}, \quad (b) \begin{pmatrix} 1 & 1 & 1 & 1 \\ b & 1 & 1 & 1 \\ b & b & 1 & 1 \\ b & b & b & 1 \end{pmatrix}, \quad (c) \begin{pmatrix} c & c & c & c & c \\ 1 & c & c & c & c \\ 1 & 1 & c & c & c \\ 1 & 1 & 1 & c & c \\ 1 & 1 & 1 & 1 & c \end{pmatrix},$$

$$(d) \begin{pmatrix} 1+a & b & c & d & e \\ a & 1+b & c & d & e \\ a & b & 1+c & d & e \\ a & b & c & 1+d & e \\ a & b & c & d & 1+e \end{pmatrix}.$$

9. Using the properties of the determinants, justify that the following matrices are not invertible

$$(a) \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{pmatrix}, \quad (b) \begin{pmatrix} 1 & 1 & -1 & -1 \\ 3 & 2 & 1 & 3 \\ 4 & 5 & 6 & 7 \\ 0 & 2 & 6 & 5 \end{pmatrix}.$$

10. For which values of the parameter c the following matrices are invertible?

$$(a) \begin{pmatrix} c & -1 \\ c & 1 \end{pmatrix}, \quad (b) \begin{pmatrix} c & 1 & 1 \\ 1 & c & -1 \\ 1 & 1 & -1 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 & 2 & c \\ -1 & 2 & -1 \\ 2 & 0 & 3 \end{pmatrix}, \quad (d) \begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & c \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

11. Using the cofactor formula, calculate the inverse matrices to the given ones, if the inverse exists:

$$(a) \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \quad (b) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad (c) \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad (d) \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix},$$

$$(e) \begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \quad (f) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

12. Solve the previous problem using the Gauss elimination method.